

Characterization of the p -Generalized Normal Distribution

Fabian Sinz^{*}, Sebastian Gerwinn, Matthias Bethge

*Max Planck Institute for Biological Cybernetics
Spemannstraße 41
72076 Tübingen, Germany*

Abstract

It is a well known fact that invariance under the orthogonal group and marginal independence uniquely characterizes the isotropic normal distribution. Here, a similar characterization is provided for the more general class of differentiable bounded L_p -spherically symmetric distributions: Every factorial distribution in this class is necessarily p -generalized normal.

Key words: characterization, generalized normal distribution, exponential power distribution, L_p -spherically symmetric distributions

1 Introduction

Kac's characterization of the normal distribution [Kac(1939)] states that the isotropic Gaussian is the only distribution in the intersection of the class of factorial distributions and the class of spherically symmetric distributions. A natural extension to the latter are the L_p -spherically symmetric distributions [Osiewalski and Steel(1993), Gupta and Song(1997)]. A random variable X is L_p -spherically symmetric distributed if it can be written as a product of two independent random variables R and U , where R is a univariate non-negative

^{*} Corresponding Author.

Email addresses: fabee@tuebingen.mpg.de (Fabian Sinz),
sgerwinn@tuebingen.mpg.de (Sebastian Gerwinn), mbethge@tuebingen.mpg.de
(Matthias Bethge).

URLs: <http://www.kyb.tuebingen.mpg.de/~fabee> (Fabian Sinz),
<http://www.kyb.tuebingen.mpg.de/~sgerwinn> (Sebastian Gerwinn),
<http://www.kyb.tuebingen.mpg.de/~mbethge> (Matthias Bethge).

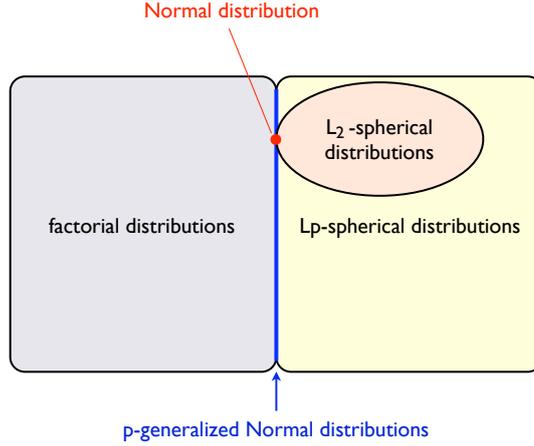


Figure 1. **Properties of the p -generalized Normal distribution** The Gaussian is the only L_2 -spherically symmetric distribution with independent marginals. Like the Gaussian, all p -generalized Normal distributions have independent marginals and the property of spherical symmetry is a special case of the L_p -spherical symmetry in this class. We prove that the p -generalized Normal distributions are the only distributions which combine these two properties simultaneously.

random variable with an arbitrary distribution and \mathbf{U} is uniformly distributed on the set $\mathcal{S}_p^{n-1} := \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p = 1\}$. Equivalently, \mathbf{X} is L_p -spherically distributed if its density has the form $g(\sum_{i=1}^n |x_i|^p)$.

This class of distributions is of great practical interest: It offers more flexibility than the spherically symmetric model, but is still easy to fit to data since it only requires estimating the univariate radial distribution. An interesting subclass is the p -generalized Normal distribution [Goodman and Kotz(1973)]

$$g\left(\sum_{i=1}^n |x_i|^p\right) = \frac{p^n}{\left(2\Gamma\left(\frac{1}{p}\right)(2\sigma^2)^{\frac{1}{p}}\right)^n} e^{-\frac{\sum_{i=1}^n |x_i|^p}{2\sigma^2}},$$

which contains the Normal distribution as a special case for $p = 2$.

Note, that the p -generalized Normal distribution is factorial with marginals from the exponential power family [Box and Tiao(1992)]. In that sense, the p -generalized Normal distribution is the analog of a Gaussian for L_p -spherically symmetric distributions. Surprisingly, to the best of our knowledge, we could not find any reference that characterizes the p -generalized Normal distribution as the only marginally independent L_p -spherically symmetric distribution. Here, we provide this characterization for the class of differentiable and bounded L_p -spherically symmetric densities.

2 Characterization

Theorem 1 Let $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be an differentiable multivariate L_p -spherically symmetric density. If g has the following properties:

- (1) $g \in C^1(\mathbb{R}^n)$
- (2) g and $\frac{\partial}{\partial x_i} g$ are bounded for all $i = 1, \dots, n$

then marginal independence, i.e. $g(\sum_{i=1}^n |x_i|^p) = \prod_{k=1}^n h(|x_k|^p)$, implies that g is p -generalized Normal, i.e. $h(|x_k|) = \frac{p}{2\Gamma(\frac{1}{p})(2\sigma^2)^{\frac{1}{p}}} \exp\left(-\frac{|x_k|^p}{2\sigma^2}\right)$.

PROOF. Let g be factorial, i.e. $g(\sum_{i=1}^n |x_i|^p) = \prod_{k=1}^n h_k(|x_k|^p)$, and let P be a permutation matrix. Since g is L_p -spherically symmetric, g is invariant under permutation of the basis elements, i.e. $g(\sum_{i=1}^n |x_i|^p) = g(\sum_{i=1}^n |y_i|^p)$ with $\mathbf{y} = P\mathbf{x}$. Choose a $\mathbf{v} \in \mathbb{R}^n$ for some $a \in \mathbb{R}$ with $v_j = a \cdot \delta_{ij}$ and P such that $P\mathbf{v} = \mathbf{w}$ with $w_j = a \cdot \delta_{kj}$. Thus,

$$\begin{aligned} g\left(\sum_{i=1}^n |v_i|^p\right) &= g\left(\sum_{i=1}^n |w_i|^p\right) \\ \Rightarrow h_i(|a|^p) \prod_{\substack{\ell=1 \\ \ell \neq i}}^n h_\ell(0) &= h_k(|a|^p) \prod_{\substack{\ell=1 \\ \ell \neq k}}^n h_\ell(0) \\ \Rightarrow h_i(a^p) &= h_k(a^p) \cdot c \quad \forall a \in \mathbb{R}_+ \quad \text{with } c = \frac{h_i(0)}{h_k(0)}. \end{aligned}$$

Since all h_i integrate to one, c must be one as well.

Note that none of the $h_i(0)$ can be zero because they can be written as $h_i(0) = \int_{\mathbb{R}^{n-1}} g\left(\sum_{k \neq i} |x_k|^p\right) dx_1 dx_2 \dots dx_{i-1} dx_{i+1} \dots dx_n$ and g a non-negative function which does not vanish everywhere. Therefore, all marginals h must have the same form, that is $g(\sum_{i=1}^n |x_i|^p) = \prod_{k=1}^n h(|x_k|^p)$.

With the particular choice of \mathbf{v} it follows $g(\sum_{i=1}^n |v_i|^p) = g(|a|^p) = h(|a|^p) \cdot h(0)^{n-1}$ or just $g(u) = h(u) \cdot h(0)^{n-1}$ by substitution $u := |a|^p$. Now, choosing $(a, b, 0, \dots, 0)^\top \in \mathbb{R}^n$ we can write

$$\begin{aligned} g(|a|^p + |b|^p) &= h(|a|^p)h(|b|^p)h(0)^{n-2} \\ &= g(|a|^p)h(0)^{1-n} \cdot g(|b|^p)h(0)^{1-n} \cdot h(0)^{n-2} \\ &= g(|a|^p)g(|b|^p)h(0)^{-n} \\ &= g(|a|^p)g(|b|^p)/g(0). \end{aligned}$$

or just $g(u + \epsilon) = g(u)g(\epsilon)/g(0)$ for all $u, \epsilon \in \mathbb{R}_+$.

Thus, we obtain

$$g(u + \varepsilon) - g(u) = \frac{g(u)}{g(0)} \cdot (g(\varepsilon) - g(0))$$

and it follows immediately

$$g'(u) = \frac{g(u)}{g(0)} g'(0)$$

Solving this differential equation uniquely yields the functional form

$$\begin{aligned} g(u) &= g(0) \exp\left(\frac{g'(0)}{g(0)} \cdot u\right) \\ &= \exp(c_1 a + c_0). \end{aligned}$$

Choosing a value for c_1 corresponds to setting the scale of the distribution. Taking into account that g must integrate to one determines c_0 and yields that h is in the exponential power family. Thus g is p -generalized Normal. \square

3 Discussion

The theorem presented in this paper provides an important theoretical insight showing that the intersection between the space of L_p -spherical distributions and the space of factorial distributions is a low-dimensional manifold known as the family of p -generalized Normal distributions. In particular, the previous characterization of the isotropic Gaussian as the only spherically symmetric factorial distribution can now be understood as the special case of the more general theorem when $p = 2$. Consequently, the range of potential applications is now extended from the special case of isotropic distributions to arbitrary L_p -spherical distributions.

An immediate consequence of the theorem concerns density estimation on empirical data. Assuming marginal independence and L_p -spherical symmetry not only implies that the marginals must be exponential power distributions but also decreases the degrees of freedom to the mean and the scale parameter of the p -generalized Normal. This shows that marginal independence is a very restrictive assumption in the class of L_p -spherical symmetric distributions which turns the infinite dimensional estimation problem of the radial distribution into a one-dimensional one.

Other consequences and applications arise from the fact that each L_p -spherically symmetric distributed random variable X has a stochastic representation $X =$

RU. By changing the radial component with the transform $\mathcal{F}_2^{-1} \circ \mathcal{F}_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, where \mathcal{F}_1 and \mathcal{F}_2 are the cumulative distribution functions of the source and the target radial distribution, respectively, one can change the distribution of \mathbf{X} within the class of L_p -spherically symmetric distributions for a particular fixed p .

From our theorem we know that there is a unique factorial distribution (up to a scale parameter) each L_p -spherically symmetric distribution can be mapped into by choosing \mathcal{F}_2 to be the c.d.f. of the p -generalized Normal distribution $\mathcal{F}_2(r) = \mathcal{F}_p(r) = \frac{\Gamma(\frac{n}{p}, \frac{r^p}{2\sigma^2})}{\Gamma(\frac{n}{p})}$, with $\Gamma(z, a)$ denoting the incomplete Γ -function.

Conversely, one can also use this relationship for efficient sampling from arbitrary L_p -spherically symmetric distributions. The idea is to first sample from a p -generalized Normal distribution and subsequently transform the radial component by setting $\mathcal{F}_1 = \mathcal{F}_p$ and setting \mathcal{F}_2 equal to the c.d.f. of the radial component \mathbf{R} of the target distribution. That is, each random vector x sampled from the p -generalized Normal is transformed by $x \mapsto \frac{(\mathcal{F}_2^{-1} \circ \mathcal{F}_p)(r)}{r} x$ with $r = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ which can be computed very fast. Furthermore, sampling from the p -generalized Normal is easy as one can sample from the univariate marginal distributions independently. Our theorem implies that the exponential power distribution is the only admissible marginal distribution with which such a sampling scheme is possible.

Finally, our theorem is also useful for constructing an independence test for L_p -spherically symmetric distributed random variables. For a given set of samples $x_1, \dots, x_m \in \mathbb{R}^n$, the radial distribution

$$q_r(r) = \frac{p r^{n-1}}{\Gamma(\frac{n}{p}) (2\sigma^2)^{\frac{n}{p}}} e^{-\frac{r^p}{2\sigma^2}}$$

of the p -generalized Normal is fitted to the radial components $r_k = (\sum_{i=1}^n |x_{ki}|^p)^{\frac{1}{p}}$, $k = 1, \dots, m$ of the data points. Afterwards, a goodness of fit test (e.g. Kolmogorov-Smirnov) can be used to test whether the x_k come from a factorial L_p -spherically symmetric distribution. Since the p -generalized Normal is the only L_p -spherically symmetric distribution with independent marginals, the test should succeed if the marginals are independent and fail if they are not. Such an independence test can be of particular interest in the context of Independent Component Analysis [Comon(1994)] in order to verify whether the data actually comply with the independence assumption underlying this method.

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